On the degree function coefficient of a simple complete ideal in dimension two

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Abstract

Let $(R, \mathfrak{M})$ be a two-dimensional regular local ring with algebraically closed residue field. Let I be a simple complete $\mathfrak{M}$-primary ideal of R and let w denote its unique Rees valuation. Then the degree function coefficient $d(I, w) = 1$.

In this note a short proof of this result is given.

Keywords: Regular local ring, simple complete ideal, degree function coefficient

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1 Introduction

In the theory of complete ideals in two-dimensional regular local rings the following results holds [2].

Let $(R, \mathfrak{M})$ be a two-dimensional regular local ring with algebraically closed residue field k. If I is a simple complete $\mathfrak{M}$-primary of R with unique Rees valuation w, then the degree function coefficient $d(I, w) = 1$.

In the first part of Remark 3.5 in [1] this result was erroneously presented as a consequence of Proposition 3.4 of that paper. To correct this we show how the above result can be obtained from [1, Propopsition 3.3]. For definitions and background information the reader is referred to [1].

Proof. First, we recall a few facts from the theory of complete ideals in two-dimensional regular local rings (with algebraically closed residue field).

- *Every complete $\mathfrak{M}$-primary ideal I of R is normal and minimally generated.*

  Indeed, I is normal since we have in R that the product of complete ideals is complete again. Since $R/\mathfrak{M}$ is infinite, there exists an element $x \in \mathfrak{M}$ such that $x\mathcal{V} = \mathfrak{M}\mathcal{V}$ for all Rees valuation rings $\mathcal{V}$ of I. Hence $R\left[\frac{\mathfrak{M}}{x}\right]$ is

contained in every Rees valuation ring $\mathcal{V}$ of I and this implies that

$$IR\Big[\frac{\mathfrak{M}}{x}\Big] \cap R = I,$$

i.e., I is *contracted* from $R\big[\frac{\mathfrak{M}}{x}\big]$. By Proposition 2.3 in [3], we know this is equivalent to

$$\mu(I) = \mathrm{ord}_R(I) + 1,$$

where $\mu(I)$ denotes the minimal number of generators of I.

In other words

$$\mu(I) = \dim_k\left(\frac{\mathfrak{M}^r}{\mathfrak{M}^{r+1}}\right) \qquad \text{with } r := \mathrm{ord}_R(I),$$

i.e., I is *minimally generated* (in the sense of Definition 3.1 in [1]).

- *Every simple complete $\mathfrak{M}$-primary ideal I of R is quasi-one-fibered.*

  If a complete $\mathfrak{M}$-primary ideal I of R is simple, then I has precisely one immediate base point, say $(R_1, \mathfrak{M}_1)$. To see this we first observe that all immediate base points of I are lying on the chart $R\big[\frac{\mathfrak{M}}{x}\big]$, where $R\big[\frac{\mathfrak{M}}{x}\big]$ is contained in every Rees valuation ring of I as in the previous point. Next, the transorm I' of I in $R\big[\frac{\mathfrak{M}}{x}\big]$ is simple and complete (see Huneke [3, Proposition 3.4 and Proposition 3.5]). This implies that I' is contained in just one prime ideal M_1 of $R\big[\frac{\mathfrak{M}}{x}\big]$. Thus $R_1 := R\big[\frac{\mathfrak{M}}{x}\big]_{M_1}$ is the unique immediate base point of I and the transform $I^{R_1} = I'_{M_1}$ is simple and complete. Thus every simple complete $\mathfrak{M}$-primary ideal I of R is *quasi-one-fibered* in the sense of Definition 1.7 in [1].

  Hence

  $$T(I) \subseteq \{v_{\mathfrak{M}}, w\}$$

  with $w \in T(I)$ (see [1, Proposition 1.5]). Here $T(I)$ denotes the set of Rees valuations of I.

  Actually we know that $T(I) = \{w\}$ because of Section 4 in [3].

  So if I is any simple complete $\mathfrak{M}$-primary ideal in a two-dimensional regular local ring $(R, \mathfrak{M})$ with algebraically closed residue field, then all the conditions of Proposition 3.3 in [1] are satisfied.

Let

$$(R, \mathfrak{M}) < (R_1, \mathfrak{M}_1) < \cdots < (R_s, \mathfrak{M}_s)$$

denote the unique quadratic sequence determined by the simple complete $\mathfrak{M}$-primary ideal I of R (according to Proposition 1.6 in [1]). By a repeated use of Proposition 3.3 in [1], it follows that

$$d(I, w) = d(I^{R_1}, w) = \ldots = d(I^{R_s}, w).$$

Since $I^{R_s} = \mathfrak{M}_s$ and $w = \mathrm{ord}_{R_s}$-valuation, we have that

$$d(I, w) = d(\mathfrak{M}_s, \mathrm{ord}_{R_s}).$$

Since ord_{R_s} is the unique Rees valuation of $\mathfrak{M}_s$, it follows from the theory of degree functions (see [4, Theorem 4.3]) that

$$e(\mathfrak{M}_s) = d(\mathfrak{M}_s, \mathrm{ord}_{R_s})\,\mathrm{ord}_{R_s}(\mathfrak{M}_s).$$

As $e(\mathfrak{M}_s) = 1$, it follows that $d(\mathfrak{M}_s, \mathrm{ord}_{R_s}) = 1$. Thus

$$d(I, w) = 1.$$

□

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